Published by Institute of Physics Publishing for SISSA/ISAS

Received: *October 15, 2004*
Accepted: *November 12, 2004*

On the equivalence of $\mathcal{N}=1$ brane worlds and geometric singularities with flux

Peter Kaste

Institute for Theoretical Physics, ETH Hönggerberg
CH–8093 Zürich, Switzerland
E-mail: kaste@itp.phys.ethz.ch

Herve Partouche

Centre de Physique Théorique, Ecole Polytechnique
F-91128 Palaiseau cedex, France
E-mail: Herve.Partouche@cpht.polytechnique.fr

Abstract: We consider Kaluza Klein reductions of M-theory on the $\mathbb{Z}_N$ orbifold of the spin bundle over S^3 along two different U(1) isometries. The first one gives rise to the familiar "large-N duality" of the $\mathcal{N}=1$ SU(N) gauge theory in which the UV is realized as the world-volume theory of N D6-branes wrapped on S^3, whereas the IR involves N units of RR flux through an S^2. The second reduction gives an equivalent version of this duality in which the UV is realized geometrically in terms of a $\mathbb{P}^1$ of A_{N-1} singularities, with one unit of RR flux through the $\mathbb{P}^1$. The IR is reached via a geometric transition and involves a single D6 brane on a lens space $S^3/\mathbb{Z}_N$ or, alternatively, a singular background $(S^2\times\mathbb{R}^4)/\mathbb{Z}_N$, with one unit of RR flux through S^2 and, localized at the singularities, an action of their stabilizer group in the U(1) RR gauge bundle, so that no massless twisted states occur. We also consider linear σ-model descriptions of these backgrounds.

Keywords: Duality in Gauge Field Theories, M-Theory, String Duality, Superstring Vacua.

 http://jhep.sissa.it/archive/papers/jhep112004033/jhep112004033.pdf

Contents

1. Introduction

The physical motivations for studying type-II string theories increased a lot when, in the course of the heterotic/type-II dualities [1, 2], it was realized that they as well can describe non-abelian gauge theories. For example, it has been shown in [3] that such theories, with $\mathcal{N} = 4$ supersymmetry in 4 dimensions, arise as type-II compactifications on $K3 \times T^2$ backgrounds, where the $K3$ develops an A-D-E type singularity (see also [4]). Similarly, by considering a compactification on a Calabi-Yau (CY) 3-fold that contains a curve of such singularities, the supersymmetry is reduced to $\mathcal{N} = 2$, [5]–[9]. In the type-IIA models, the non-abelian gauge degrees of freedom arise nonperturbatively from D2-branes wrapped on the vanishing 2-cycles of a singularity that locally looks like $\mathbb{R}^4/\mathbb{Z}_N \times S^2$ in the case of the pure SU(N) gauge theory [3]. At this stage, a natural strategy to further reduce the supersymmetry to $\mathcal{N} = 1$ would be to add Ramond-Ramond (RR) flux on S^2. This is precisely what we want to consider in this paper.

In the meanwhile, however, attention has focused on another realization of non-abelian gauge theories in type-II strings, namely the effective world-volume theory on D-branes generated by open strings ending on them. $\mathcal{N} = 1$ gauge theories in 4 dimensions can be described this way by considering the type-IIA string on a smooth CY manifold with 3 directions of N coincident D6-branes wrapped on a compact special lagrangian submanifold, while the other 3+1 world volume directions fill a transverse $\mathbb{R}^{3,1}$. A simple local example is given by N D6-branes wrapped on the special lagrangian S^3 of the non-compact deformed conifold. In the classical large radius limit, in which stringy α'-corrections are suppressed, this describes the pure SU(N) super Yang-Mills (SYM) theory in the UV, with the gauge coupling constant being related to the volume of the S^3 roughly as $1/g^2_{YM} \sim \mathrm{vol}(S^3)$. In the quantum parameter space, due to the complexification with the C-field or B-field respectively, one can follow the system smoothly through a geometrical conifold transition

JHEP11(2004)033

in the course of which the S^3 shrinks to zero size and gets replaced by an S^2. After the transition, the system is described by N units of RR flux through the S^2 of the resolved conifold and is associated to the (confining) IR of the gauge theory. The effect of the flux is to partially break supersymmetry from $\mathcal{N}=2$ to $\mathcal{N}=1$ [10, 11] by adding a magnetic Fayet-Iliopoulos term [12, 13] for the closed string U(1) associated to the homology class of the S^2. This $S^3 \to S^2$ transition, with branes on the one hand side and flux on the other, is the prime example of the "large-N duality" of [14]. As there are two birationally equivalent small resolutions of the singular conifold, there are actually two equivalent realizations of the IR phase, with their S^2's related by a flop transition. In [15], it was shown how to derive the connection between the UV and IR phases from Kaluza-Klein (KK) reduction of M-theory on a background of G_2 holonomy, which is an orbifold of the spin bundle over S^3 that we denote by $\mathrm{Spin}(S^3)/\mathbb{Z}_N$. The complex one-dimensional quantum parameter space of these models indeed connects three classical limits [16]: One of them associated to the UV and the other two to the IR of the SU(N) gauge theory.[1]

In this paper, we shall derive and complete an alternative type-IIA picture of the same large-N duality, already mentioned in [18], by reducing the M-theory background along a different U(1) isometry. In this version, the UV of the SU(N) SYM theory is realized in type-IIA by a $\mathbb{P}^1$ of A_{N-1} singularities, with one unit of RR flux through the $\mathbb{P}^1$. This set up generalizes the spontaneous breaking of $\mathcal{N}=2$ to $\mathcal{N}=1$ of [10, 11] to the non-abelian case of U(N). It involves a magnetic Fayet-Iliopoulos term for the diagonal U(1) associated to the homology class of the $\mathbb{P}^1$. The corresponding field theory analysis extending the model of [12, 13] has appeared in the recent paper [19]. In type IIA, the gauge coupling of such models is again related to the volume of the base $\mathbb{P}^1$ as $1/g^2_{YM} \sim \mathrm{vol}(\mathbb{P}^1)$. The IR physics involves after a conifold transition a single D6-brane wrapped on a lens space $S^3/\mathbb{Z}_N$. The third branch, also associated to the IR, involves a singular background of topology $(S^2 \times \mathbb{R}^4)/\mathbb{Z}_N$, with one unit of RR flux through S^2, and can be reached geometrically from the UV phase by a flop transition. For N even, the orbifold group acts as $\mathbb{Z}_2 \times \mathbb{Z}_{N/2}$, whose $\mathbb{Z}_2$ subgroup fixes an $S^2/\mathbb{Z}_{N/2}$ of A_1 singularities, while for any N the north and south poles of the S^2 (times the origin in $\mathbb{R}^4$) are fixed under the whole $\mathbb{Z}_N$. There are, however, no massless states associated to these singularities since, at their locus, there is a free action of their stabilizer group within the U(1) gauge fiber associated to the RR one-form potential. This is the same mechanism that removes massless states from twisted sectors in the type-IIA dual of the six-dimensional CHL compactification [20].

The paper is organized as follows. Section 2 is devoted to the KK reductions of the three classical phases of the M-theory background along the two different U(1) isometries. In particular, it clarifies how the original large-N duality involving N D6-branes is on equal footing with the picture based on the A_{N-1} singularity with flux in type IIA. For this second point of view, we study in section 3 how the three phases are connected via conifold or flop transitions. Finally, in section 4 we present linear σ-model descriptions of the six-dimensional type-IIA backgrounds, together with their lifts to seven dimensions associated to the G_2 holonomy spaces occurring in M-theory.

[1] Another description of these phases in terms of a D6-brane in $\mathbb{C}^2/\mathbb{Z}_N \times \mathbb{C}$ can be found in [17].

JHEP11(2004)033

2. KK reductions of G_2 holonomy spaces to type IIA

We start by considering the geometrical realization of the $\mathcal{N}=1$ pure SU(N) gauge theory in M-theory [15, 21]. The relevant background is the G_2 holonomy manifold Spin(S^3) modded by $\mathbb{Z}_N$. Classically, the associated moduli space is composed of three branches [16].[2] In one of them, the seven-dimensional orbifold is singular and the massless spectrum contains the non-abelian SU(N) vector multiplets. In the last two branches, the $\mathbb{Z}_N$ orbifold acts freely and there are no massless non-abelian gauge bosons anymore. In the description of [16], the SU(2)^3 isometry of the G_2 holonomy metric on Spin(S^3) is explicit and the orbifold group $\mathbb{Z}_N$ is taken to be a subgroup of the diagonal U(1) in the first of these SU(2)'s. The KK reduction along this U(1) subgroup containing the $\mathbb{Z}_N$ was considered in [15] and allowed to associate the moduli space branches to the phases involved in the large-N duality in type IIA of [14]. Performing now the KK reduction along another U(1) isometry that does not contain the $\mathbb{Z}_N$, we derive and complete the alternative version of the large-N duality introduced in [18].

The spin bundle over S^3, Spin(S^3), is a 7-dimensional manifold that is asymptotically a cone over $S^3\times S^3$. A convenient description of the $S^3\times S^3$ base is in terms of a coset $\mathrm{SU}(2)^3/\,\mathrm{SU}(2)_D$, where $\mathrm{SU}(2)_D$ is the diagonal SU(2) factor acting on the right:

$$(g_1,g_2,g_3)\in \mathrm{SU}(2)^3\,,$$

such that

$$(g_1,g_2,g_3)\equiv(g_1h,g_2h,g_3h)\,,\qquad h\in\mathrm{SU}(2)_D\,. \tag{2.1}$$

The 7-manifold is then constructed by replacing one of the $S^3\simeq\mathrm{SU}(2)$ factors by $\mathbb{R}^4$, i.e. one allows one of the g_i's ($i=1,2,3$) to take values in the set of 2×2 matrices of the form

$$g=\begin{pmatrix} Z & iZ' \\ i\bar{Z}' & \bar{Z}\end{pmatrix},\qquad \text{where } Z,Z'\in\mathbb{C}, \tag{2.2}$$

but without constraining the determinant to be one. By choosing which of the three S^3 factors is replaced, one obtains this way three isomorphic 7-manifolds [16] that admit asymptotically conical metrics of G_2 holonomy [24, 25].

With respect to these metrics, there is an $\mathrm{SU}(2)_{1,L}\times\mathrm{SU}(2)_{2,L}\times\mathrm{SU}(2)_{3,L}$ group of isometries acting on these manifolds that is realized by left multiplication,

$$(g_1,g_2,g_3)\to(h_1g_1,h_2g_2,h_3g_3)\,,$$

where

$$(h_1,h_2,h_3)\in\mathrm{SU}(2)_{1,L}\times\mathrm{SU}(2)_{2,L}\times\mathrm{SU}(2)_{3,L}\,. \tag{2.3}$$

[2]The branches for similar models based on spaces of the form $(\mathrm{CY}\times S^1)/\mathbb{Z}_2$ have also been considered: Examples involving compact CY's are treated in [22], while the case where the CY is the deformed conifold is introduced in [23].

At this stage, the three manifolds are smooth. In order to describe an SU(N) gauge theory, one considers a $\mathbb{Z}_N$ orbifold of them. This $\mathbb{Z}_N$ is chosen as a discrete subgroup of $\mathrm{SU}(2)_{1,L}$, its generator being

$$\xi = \begin{pmatrix} e^{2\pi i/N} & 0 \\ 0 & e^{-2\pi i/N} \end{pmatrix} . \tag{2.4}$$

Note that the left action ξg amounts to $(Z, Z') \to e^{2\pi i/N}(Z, Z')$ on the coordinates in (2.2), whereas a right action $g\xi$ would act as $(Z, Z') \to (e^{2\pi i/N} Z, e^{-2\pi i/N} Z')$.

We now describe the three spaces $\mathcal{G}_I$, $\mathcal{G}_{II}$ and $\mathcal{G}_{III}$ obtained this way and reduce them along two different U(1) isometry groups, the diagonal

$$\mathrm{U}(1)_{1,L} \subset \mathrm{SU}(2)_{1,L} \qquad \text{and} \qquad \mathrm{U}(1)_{2,L} \subset \mathrm{SU}(2)_{2,L} \,, \tag{2.5}$$

which are relevant to derive the dualities of [14] and [18], respectively.[3] Some explicit expressions for the G_2 metrics and the relevant isometry actions are given in appendix A.

- Fixing the gauge $g_3 \equiv 1$ in eq. (2.1), the space $\mathcal{G}_I$ can be parametrized as[4]

  $$\mathcal{G}_I : \; [\xi^k g_1, g_2, 1] \,, \; g_1 \in \mathbb{R}^4 \,, \; g_2 \in \mathrm{SU}(2) \,, \tag{2.6}$$

  where it is understood that points with different integer values of k are identified. Topologically, this space is the orbifold $\mathbb{R}^4/\mathbb{Z}_N \times S^3$ and there are massless vector multiplets occurring in M-theory. This describes the SU(N) SYM theory in the UV. To perform the KK reduction of $\mathbb{R}^4/\mathbb{Z}_N$ along $\mathrm{U}(1)_{1,L}$, think of $\mathbb{R}^4/\mathbb{Z}_N$ as the disjoint union of lens spaces $S^3/\mathbb{Z}_N$ of arbitrary radii $\rho \in \mathbb{R}_{\geq 0}$. As recalled in appendix B, the Hopf reduction of each such lens space is an S^2 of radius $f(\rho)$, where $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is one-to-one and onto, and with N units of RR flux through it. Letting ρ vary, one sees that $(\mathbb{R}^4/\mathbb{Z}_N)/\,\mathrm{U}(1)_{1,L}$ is $\mathbb{R}^3$ with a magnetic source of N units for the RR two-form field strength at the origin, i.e. there are N D6-branes at the vanishing locus of the orbit of $\mathrm{U}(1)_{1,L}$, [27]. This type-IIA background is the deformed conifold[5] T^*S^3 with N D6-branes wrapped on S^3.

  In the same spirit, we can consider the KK reduction of $\mathcal{G}_I$ along $\mathrm{U}(1)_{2,L}$. Since $\mathrm{U}(1)_{2,L}$ acts on $S^3 \simeq \mathrm{SU}(2)$ only, the $\mathbb{R}^4/\mathbb{Z}_N$ factor remains as it is. Seen as an Hopf fibration, the reduction of S^3 gives rise to a two-sphere with one unit of RR flux through S^2 (see appendix B with $N = 1$). Topologically, the type-IIA background is thus a $\mathbb{P}^1$ of A_{N-1} singularities, $\mathbb{R}^4/\mathbb{Z}_N \times S^2$, that we shall denote $\mathcal{C}_I$. It is the

[3]As a consequence of the fact that asymptotically the G_2 metric we consider is conical, the string coupling in the associated type-IIA models is unbounded. In [26], another metric of G_2 holonomy is given that is not asymptotically conical and has a reduced isometry group $\mathrm{U}(1)_{1,L} \times \mathrm{SU}(2)_{2,L} \times \mathrm{SU}(2)_{3,L} \times \mathbb{Z}_2$. Considering the orbifold by $\mathbb{Z}_N \subset \mathrm{U}(1)_{1,L}$, the KK reduction along $\mathrm{U}(1)_{1,L}$ gives rise to type-IIA with N D6-branes on S^3 with a bounded dilaton, while a reduction along $\mathrm{U}(1)_{2,L} \subset \mathrm{SU}(2)_{2,L}$ gives our alternative realization as a singularity with flux in which, however, the dilaton is unbounded.

[4]Our convention is to use square brackets when referring to representatives of equivalence classes.

[5]The metric on this space is of course not the Calabi-Yau metric due to the back-reaction to the branes. The same comment will apply to all conifold spaces in the presence of branes or fluxes that we shall encounter. The respective metrics can easily be derived by using the expressions given in appendix A.

JHEP11(2004)033

resolved conifold with one unit of RR flux on the S^2 and a $\mathbb{Z}_N$ orbifold action on the $\mathbb{R}^4$ fiber.

- Again fixing $g_3 \equiv 1$ thanks to eq. (2.1), the manifold $\mathcal{G}_{II}$ can be defined as

$$\mathcal{G}_{II}: \ [\xi^k g_1, g_2, 1], \ g_1 \in \mathrm{SU}(2), \ g_2 \in \mathbb{R}^4, \tag{2.7}$$

where an identification of the points with different values of k is understood. There are no fixed points under the orbifold and topologically the manifold is $S^3/\mathbb{Z}_N \times \mathbb{R}^4$, where $S^3/\mathbb{Z}_N$ is a lens space. There is no massless non-abelian gauge multiplet as expected for the SU(N) SYM theory in the IR. The KK reduction along $\mathrm{U}(1)_{1,L}$ acts only on the $S^3/\mathbb{Z}_N$ factor and gives a two-sphere with N units of RR flux through it, as reviewed in appendix B. Altogether, the topology is thus $S^2 \times \mathbb{R}^4$ and corresponds to a type-IIA string on the resolved conifold with N units of RR flux through the S^2. The presence of flux has the effect to partially break $\mathcal{N}=2$ to $\mathcal{N}=1$. This effect is a particular case of the general set up of [10, 11] that embeds in string theory the field theory mechanism of [12, 13] for rigid supersymmetry.

For the reduction of $\mathcal{G}_{II}$ along $\mathrm{U}(1)_{2,L}$, only the $\mathbb{R}^4$ factor is concerned and gives $\mathbb{R}^3$, with a single D6-brane at the origin that wraps the lens space. One thus obtains a background whose topology is $S^3/\mathbb{Z}_N \times \mathbb{R}^3$, the deformed conifold whose base is modded by a freely acting $\mathbb{Z}_N$, with one D6-brane wrapped on the lens space. We shall call this space $\mathcal{C}_{II}$. Since the D6-brane carries one unit of magnetic charge w.r.t. the type-IIA RR one-form gauge potential, there is conservation of the RR charge between the present background and $\mathcal{C}_I$ with one unit of flux.

- Finally, in the gauge $g_2 \equiv 1$ in eq. (2.1), the manifold $\mathcal{G}_{III}$ can be defined as

$$\mathcal{G}_{III}: \ [\xi^k g_1, 1, g_3], \ g_1 \in \mathrm{SU}(2), \ g_3 \in \mathbb{R}^4. \tag{2.8}$$

It is isometric to $\mathcal{G}_{II}$ via the exchange $g_2 \leftrightarrow g_3$ and thus associated to the IR of the SYM theory as well. The KK reduction along $\mathrm{U}(1)_{1,L}$ gives then again the resolved conifold with N units of RR flux through the $\mathbb{P}^1$. The $\mathbb{P}^1$'s obtained by $\mathrm{U}(1)_{1,L}$ reduction of $\mathcal{G}_{II}$ and $\mathcal{G}_{III}$ are related by a flop.

To reduce $\mathcal{G}_{III}$ along $\mathrm{U}(1)_{2,L}$, it is convenient to choose instead the gauge $g_1 \equiv 1$ in eq. (2.1), so that we have:

$$\mathcal{G}_{III}: \ [1, g_2\xi^{-k}, g_3\xi^{-k}], \ g_2 \in \mathrm{SU}(2), \ g_3 \in \mathbb{R}^4, \tag{2.9}$$

where points with different values for k are identified. In this parametrization, the freely acting $\mathbb{Z}_N$ acts simultaneously from the *right* on the S^3 and $\mathbb{R}^4$ factors. As shown in appendix C, after KK reduction to type IIA, the background topology is $(S^2 \times \mathbb{R}^4)/\mathbb{Z}_N$, with one unit of RR flux through S^2. It corresponds to an orbifold of the resolved conifold we shall call $\mathcal{C}_{III}$. In order to discuss the orbifold action in type IIA, we distinguish the cases of odd respectively even N, parametrizing the $\mathbb{R}^4$ by two complex coordinates Z_3 and Z_3' in both cases. For odd N, the orbifold group acts on the two-sphere as rotations around the axis through the north and south poles by angles $2\pi k/N$, combined with an action

$$(Z_3, Z_3') \equiv \left((-1)^k e^{i\pi k/N} Z_3, (-1)^k e^{-i\pi k/N} Z_3'\right)$$

on $\mathbb{R}^4$, for $k = 0, \ldots, N-1$. This geometrical orbifold action fixes two points of the six-dimensional space, namely the north and south poles of the S^2 (times the origin in the $\mathbb{R}^4$-fiber). Localized at these two fixed points, however, there is an embedding of the $\mathbb{Z}_N$ orbifold action within the U(1) gauge fiber associated with the RR one-form potential. The orbifold doesn't act on the gauge fibers over any other non-fixed point. For even N, the orbifold group acts as $\mathbb{Z}_2 \times \mathbb{Z}_{N/2}$, where the $\mathbb{Z}_2$ acts trivially on the two-sphere and the $\mathbb{Z}_{N/2}$ by rotations around the axis through the north and south poles by angles $4\pi l/N$, whereas the $\mathbb{Z}_2 \times \mathbb{Z}_{N/2}$ acts on the $\mathbb{R}^4$ as

$$(Z_3, Z_3') \equiv \left(e^{2i\pi(l+\frac{N}{2}\alpha)/N} Z_3, e^{-2i\pi(l+\frac{N}{2}\alpha)/N} Z_3'\right) = \left(e^{2i\pi k/N} Z_3, e^{-2i\pi k/N} Z_3'\right),$$

with $k = l + \frac{N}{2}\alpha$, where $l = 0, \ldots, \frac{N}{2} - 1$ and $\alpha = 0, 1$. Note that the generator of $\mathbb{Z}_2$ always leads to an $S^2/\mathbb{Z}_{N/2}$ of A_1 singularities given by $Z_3 = Z_3' = 0$. The poles of the base (times the origin in the fiber) are always fixed by the whole $\mathbb{Z}_N$. Again there is a localized embedding of the $\mathbb{Z}_2$ (respectively $\mathbb{Z}_N$) orbifold action in the U(1) gauge symmetry associated with the RR one-form potential precisely at the $\mathbb{Z}_2$ (respectively $\mathbb{Z}_N$)-fixed loci of the geometrical orbifold action.

In a nutshell, the free $\mathbb{Z}_N$ orbifold action on $\mathcal{G}_{III}$ reduces to a geometrical $\mathbb{Z}_N$ orbifold action on $\mathcal{C}_{III}$ with $\mathbb{Z}_N$ and $\mathbb{Z}_2$-fixed loci (the latter for even N). The type-IIA theory, however, "remembers" that the action on $\mathcal{G}_{III}$ was free by a localized free orbifold action of the stabilizer subgroups $\mathbb{Z}_N$ and $\mathbb{Z}_2$ on the RR U(1) gauge fiber at the respective fixed point loci. These additional RR gauge twists localized at the fixed point set of the geometric orbifold actions are responsible for removing the massless states that would otherwise occur through their twisted sectors. For $N = 2$, it is these RR gauge twists that make the difference between the otherwise identical models $\mathcal{C}_I$ and $\mathcal{C}_{III}$, where the former has an SU(2) gauge symmetry whereas the latter does not. This is exactly the same mechanism that reduced the gauge symmetry in the type-IIA dual of the six-dimensional CHL compactification [20]. As in that case, the precise working of this mechanism from the type-IIA perspective is not well understood, due to the lack of a CFT description of these nontrivial RR backgrounds. It looks very much like introducing an additional circle corresponding to the RR U(1) fiber (the M-theory circle) with an ordinary shift orbifold action on it. The result of removing massless states, however, can be inferred from the fact that the orbifold action is free in the dual M-theory realization $\mathcal{G}_{III}$ due to the shift in the M-theory circle.

3. Transitions from the type-IIA point of view

The KK reductions of the G_2 holonomy spaces $\mathcal{G}_I$, $\mathcal{G}_{II}$ and $\mathcal{G}_{III}$ along $\mathrm{U}(1)_{2,L}$ give rise to the 6-dimensional spaces $\mathcal{C}_I$, $\mathcal{C}_{II}$ and $\mathcal{C}_{III}$, together with RR flux or a D6-brane. In this section are going to discuss the geometric transitions between the different type-IIA geometries.

JHEP11(2004)033

$\mathcal{C}_I$ is the resolved conifold with a $\mathbb{Z}_N$ modding action on the fiber. Let us introduce four complex variables z_i ($i = 1, \ldots, 4$) and homogeneous coordinates $[\xi_1, \xi_2]$ for the base $\mathbb{P}^1$. The resolved conifold can then be defined as

$$\begin{cases} (z_1 + iz_2)\xi_1 - (z_3 + iz_4)\xi_2 = 0 \\ (z_3 - iz_4)\xi_1 + (z_1 - iz_2)\xi_2 = 0\,. \end{cases} \tag{3.1}$$

$\mathcal{C}_I$ is obtained under a discrete isometry identification that we can define as

$$\big((z_1 \pm iz_2), (z_3 \pm iz_4)\big) \equiv \left(e^{\pm 2\pi i/N}(z_1 \pm iz_2), e^{\pm 2\pi i/N}(z_3 \pm iz_4)\right). \tag{3.2}$$

In addition, in the type-IIA background, there is one unit of RR flux on the $\mathbb{P}^1$ so that we have massless $\mathcal{N} = 1$ SU(N) vector multiplets arising from D2-branes wrapped on the vanishing cycles of the A_{N-1} singularity at $z_{1,\ldots,4} = 0$. Both from an M-theory and physical point of view, we know that there are no massless scalars in the adjoint representation of the gauge group so that there is no analog of the Coulomb branch of the $\mathcal{N} = 2$ SU(N) theory. In type IIA, the blow up parameters of the A_{N-1} singularity are part of complex scalars, which are moduli in the $\mathcal{N} = 2$ case. In presence of flux, there is a non trivial superpotential, whose effect is to lift these flat directions. Classically, the blow down geometry is then frozen and SU(N) is not broken.

To see explicitly the transition from $\mathcal{C}_I$ to $\mathcal{C}_{II}$, we perform the conifold transition from the resolved conifold to the deformed conifold. First of all, since $[\xi_1, \xi_2]$ are projective coordinates, they can't vanish simultaneously and we can replace the first eq. of (3.1) by the vanishing determinant of the system

$$(z_1 + iz_2)(z_1 - iz_2) + (z_3 + iz_4)(z_3 - iz_4) = 0\,. \tag{3.3}$$

When the volume of the base $\mathbb{P}^1$ vanishes, the second equation of (3.1) that determines $\xi_{1,2}$ is then useless and we can omit it. Thus, we are left with eq. (3.3) that describes the conifold, which is singular at the origin $z_{1,\ldots,4} = 0$. The deformed conifold T^*S^3 is obtained by adding a constant to the r.h.s.

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = \mu\,, \tag{3.4}$$

where μ can be chosen to be real and positive without loss of generality. The manifold $\mathcal{C}_{II}$ is then obtained by taking into account the identification (3.2). To see explicitly that there is a lens space base $S^3/\mathbb{Z}_N$ in $\mathcal{C}_{II}$, one identifies the S^3 in T^*S^3 as the set of points satisfying eq. (3.4) with real $z_{1,\ldots,4}$ and rewrite eq. (3.2) in the equivalent form

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \equiv \begin{pmatrix} \cos\left(\frac{2\pi}{N}\right) & -\sin\left(\frac{2\pi}{N}\right) \\ \sin\left(\frac{2\pi}{N}\right) & \cos\left(\frac{2\pi}{N}\right) \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \qquad \begin{pmatrix} z_3 \\ z_4 \end{pmatrix} \equiv \begin{pmatrix} \cos\left(\frac{2\pi}{N}\right) & -\sin\left(\frac{2\pi}{N}\right) \\ \sin\left(\frac{2\pi}{N}\right) & \cos\left(\frac{2\pi}{N}\right) \end{pmatrix} \begin{pmatrix} z_3 \\ z_4 \end{pmatrix}. \tag{3.5}$$

As seen in section 2, the full type-IIA background in this phase contains a D6-brane wrapped on the base lens space. The D6-brane has the effect to break supersymmetry to $\mathcal{N} = 1$ on its world-volume. The closed string U(1) vector multiplet associated to the base $\mathbb{P}^1$ of $\mathcal{C}_I$ is dual to the open string U(1) on the world volume of the D6-brane.

JHEP11(2004)033

Finally, we would like to connect the moduli space of $\mathcal{C}_{III}$ to the two previous branches associated to $\mathcal{C}_I$ and $\mathcal{C}_{II}$. The equations of the resolved conifold (3.1) imply eq. (3.3) and the definition of the blow up $\mathbb{P}^1$ parametrized by $[\xi_1, \xi_2]$ at the origin was chosen to be

$$\frac{z_1 + iz_2}{z_3 + iz_4} = -\frac{z_3 - iz_4}{z_1 - iz_2} = \frac{\xi_2}{\xi_1}\,. \tag{3.6}$$

Performing the flop transition on $\mathbb{P}^1$ amounts to defining instead projective coordinates $[\zeta_1, \zeta_2]$ as

$$-\frac{z_1 + iz_2}{z_3 - iz_4} = \frac{z_3 + iz_4}{z_1 - iz_2} = \frac{\zeta_2}{\zeta_1}\,. \tag{3.7}$$

After the flop transition, the resolved conifold is then taking the form [28]

$$\left\{ \begin{array}{l} \quad (z_1 + iz_2)\zeta_1 + (z_3 - iz_4)\zeta_2 = 0 \\ -(z_3 + iz_4)\zeta_1 + (z_1 - iz_2)\zeta_2 = 0\,. \end{array} \right. \tag{3.8}$$

To obtain a full definition of $\mathcal{C}_{III}$, we consider the $\mathbb{Z}_N$ orbifold on the variables $z_{1,\dots,4}$ given in eq. (3.2) and extend its action to $[\zeta_1, \zeta_2]$ such that it is a discrete isometry group of the resolved conifold (3.8),

$$\left\{ \begin{array}{l} \big((z_1 \pm iz_2), (z_3 \pm iz_4)\big) \equiv \big(e^{\pm 2\pi i/N}(z_1 \pm iz_2), e^{\pm 2\pi i/N}(z_3 \pm iz_4)\big) \\ [\zeta_1, \zeta_2] \equiv [e^{-2\pi i/N}\zeta_1, e^{2\pi i/N}\zeta_2]\,. \end{array} \right. \tag{3.9}$$

The inhomogeneous coordinate on the chart $\zeta_1 \neq 0$ is ζ_2/ζ_1 and the restriction of the orbifold action to the base $\mathbb{P}^1$ is $[1, \zeta_2/\zeta_1] \equiv [1, e^{4\pi i/N}\zeta_2/\zeta_1]$. This corresponds to a $\mathbb{Z}_{N/2}$ action for N even, and to a $\mathbb{Z}_N$ action for N odd. The fixed points on $\mathcal{C}_{III}$ satisfy $z_{1,\dots,4} = 0$, i.e. are sitting on the base. For odd N, in the chart $\zeta_1 \neq 0$, only the north pole $[1, 0]$ is fixed, while in the chart $\zeta_2 \neq 0$, only the south pole $[0, 1]$ is fixed. For even N, not only these poles are fixed under $\mathbb{Z}_N$, but the full base $\mathbb{P}^1$ is invariant under the $\mathbb{Z}_2$ subgroup generated by the $(k = N/2)$'th power of the above generator. Thus, for any N, the peculiarities of the restriction of the $\mathbb{Z}_N$ action to the base together with the fixed points set on $\mathcal{C}_{III}$ are precisely what was found in appendix C and summarized in section 2.

4. Linear σ-model descriptions

In this section, we first give linear σ-model descriptions of the type-IIA backgrounds $\mathcal{C}_I$ and $\mathcal{C}_{III}$, when the RR flux is not included. Then, a way to take into account this flux is actually to derive linear σ-model descriptions of the full G_2 holonomy spaces $\mathcal{G}_I$, $\mathcal{G}_{II}$ and $\mathcal{G}_{III}$.

$\mathcal{C}_I$ and $\mathcal{C}_{III}$ are orbifolds of the resolved conifold. In order to describe them in terms of a linear σ-model, one considers a two-dimensional $\mathcal{N} = (2, 2)$ supersymmetric abelian gauge theory whose low energy field configuration in a certain limit of parameter space sweeps out the desired CY, [29]. For the model associated with the resolved conifold, one introduces a U(1) gauge field coupled to four chiral ones, whose charges are $Q = (1, -1, -1, 1)$. Their scalar components $x_{1,\dots,4}$ are thus subject to the U(1) gauge equivalence

$$(x_1, x_2, x_3, x_4) \equiv \left(e^{i\lambda}x_1, e^{-i\lambda}x_2, e^{-i\lambda}x_3, e^{i\lambda}x_4\right), \tag{4.1}$$

where λ is real, and the vanishing D-term condition

$$|x_1|^2 + |x_4|^2 - |x_2|^2 - |x_3|^2 = t\,, \tag{4.2}$$

where t is a Fayet-Iliopoulos term.

To make contact with the description of eqs. (3.1) and (3.8), we identify the z_i variables with the gauge invariant combinations of the x_i's under (4.1) [30]

$$z_1 + iz_2 = x_1x_2\,, \qquad z_1 - iz_2 = x_3x_4\,, \qquad -(z_3 - iz_4) = x_1x_3\,, \qquad z_3 + iz_4 = x_2x_4\,. \tag{4.3}$$

Note that these relations are consistent with eq. (3.3). From eq. (3.3), one can see an explicit $\mathrm{U}(1) \times \mathrm{U}(1)$ isometry group acting as

$$(z_1 \pm iz_2) \to e^{\pm i\alpha}(z_1 \pm iz_2)\,, \qquad (z_3 \pm iz_4) \to e^{\pm i\beta}(z_3 \pm iz_4)\,, \tag{4.4}$$

that can be lifted to $\xi_{1,2}$ or $\zeta_{1,2}$ by consistency with eqs. (3.6) and (3.7). In terms of the original variables $x_{1,\dots,4}$, this isometry group amounts to

$$\big(x_1, x_2, x_3, x_4\big) \to \Big(e^{i(\alpha-\beta)/2}x_1, e^{i(\alpha+\beta)/2}x_2, e^{-i(\alpha+\beta)/2}x_3, e^{-i(\alpha-\beta)/2}x_4\Big)\,. \tag{4.5}$$

Now, the linear σ-models of $\mathcal{C}_I$ and $\mathcal{C}_{III}$ are obtained by taking into account the $\mathbb{Z}_N$ subgroup action eq. (3.2), whose generator corresponds to $\alpha = \beta = 2\pi/N$:

$$\mathbb{Z}_N:\ (x_1, x_2, x_3, x_4) \equiv \Big(x_1, e^{2\pi i/N}x_2, e^{-2\pi i/N}x_3, x_4\Big)\,, \tag{4.6}$$

which gives additional identifications between equivalence classes of the $\mathrm{U}(1)$ action (4.1).

For $t > 0$ in eq. (4.2), one sees that $(x_1, x_4) \neq (0,0)$ and the whole space can be covered by two charts $x_1 \neq 0$ and $x_4 \neq 0$. For $x_1 \neq 0$, we can fix representatives of the equivalence classes of (4.1) by using the $\mathrm{U}(1)$ action to remove the phase of x_1, i.e. in this chart representatives have the form $[r_1, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}]$, with $r_1 > 0$. The $\mathbb{Z}_N$ action (4.6) acts on these representatives as

$$\Big[r_1, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}\Big] \equiv \Big[r_1, e^{2\pi i/N}x_2^{(1)}, e^{-2\pi i/N}x_3^{(1)}, x_4^{(1)}\Big]\,. \tag{4.7}$$

The fixed points $[\sqrt{t - |x_4^{(1)}|^2}, 0, 0, x_4^{(1)}]$ parametrize the base $\mathbb{P}^1$ minus its north pole. Similarly, in the chart $x_4 \neq 0$, fixing the representatives of the $\mathrm{U}(1)$ equivalence classes to be of the form $[x_1^{(4)}, x_2^{(4)}, x_3^{(4)}, r_4]$ with $r_4 > 0$, the $\mathbb{Z}_N$ action looks analogous to (4.7) and fixes the base $\mathbb{P}^1$ minus its south pole. Altogether, there is a $\mathbb{P}^1$ of A_{N-1} singularities in the σ-model description. Therefore, the phase $t > 0$ corresponds to $\mathcal{C}_I$.

For $t < 0$ in eq. (4.2), one has instead $(x_2, x_3) \neq (0,0)$ and we can cover the space with two charts $x_2 \neq 0$ and $x_3 \neq 0$. For $x_2 \neq 0$, fixing the representatives of the $\mathrm{U}(1)$ classes to have the form $[x_1^{(2)}, r_2, x_3^{(2)}, x_4^{(2)}]$ with $r_2 > 0$, the $\mathbb{Z}_N$ action (4.6) amounts to

$$\Big[x_1^{(2)}, r_2, x_3^{(2)}, x_4^{(2)}\Big] \equiv \Big[e^{2\pi i/N}x_1^{(2)}, r_2, e^{-4\pi i/N}x_3^{(2)}, e^{2\pi i/N}x_4^{(2)}\Big]\,. \tag{4.8}$$

For N even, we find again the $\mathbb{Z}_{N/2}$ action on the base $\mathbb{P}^1$ minus the north pole $\Big[0, \sqrt{|t| - |x_3^{(2)}|^2}, x_3^{(2)}, 0\Big]$, while for N odd, this action is of order N. For odd N, only the south pole $[0, \sqrt{|t|}, 0, 0]$ is fixed under $\mathbb{Z}_N$, while for even N, in addition the whole $\mathbb{P}^1$ minus the north pole is fixed under a $\mathbb{Z}_2$ subgroup. Similarly, in the chart $x_3 \neq 0$, one finds the same features, with the roles of the north and south poles inverted. After gluing together the two charts, the σ-model in the phase $t < 0$ appears thus to be associated to $\mathcal{C}_{III}$.

We would like now to consider the effect of adding flux. This can be done by constructing σ-model descriptions of $\mathcal{G}_I$ and $\mathcal{G}_{III}$, the purely geometrical backgrounds which are dual in M-theory to the type-IIA compactifications on $\mathcal{C}_I$ and $\mathcal{C}_{III}$ in presence of RR flux. We shall use the formalism introduced in [30], where 7-dimensional G_2 manifolds are described by linear σ-models. However, we shall have in addition to implement the orbifold action in this formalism.

As a starting point, let us recall the σ-model of Spin(S^3) seen as a lift of the resolved conifold with one unit of RR flux through S^2. To this end, one considers the original scalars $x_1,\ldots,x_4$ subject to the D-term constraint eq. (4.2), together with an additional 2π-periodic real variable ϕ, where the U(1) modding action is now [30]

$$(x_1, x_2, x_3, x_4; \phi) \equiv (e^{i\lambda}x_1, e^{-i\lambda}x_2, e^{-i\lambda}x_3, e^{i\lambda}x_4; \phi + \lambda)\,. \tag{4.9}$$

Choosing $\lambda = -\phi$, the representatives of the equivalence classes of this U(1) action are taken to be $[x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, x_4^{(0)}; 0]$. The linear σ-model of Spin(S^3) is then described in terms of the real D-term constraint on the gauge fixed complex variables $x_i^{(0)}$,

$$|x_1^{(0)}|^2 + |x_4^{(0)}|^2 - |x_2^{(0)}|^2 - |x_3^{(0)}|^2 = t\,, \tag{4.10}$$

and the resulting space is thus 7-dimensional. For $t > 0$, it is an $\mathbb{R}^4$ fibration parametrized by $x_2^{(0)}$ and $x_3^{(0)}$, over an S^3 base spanned by $x_1^{(0)}$ and $x_4^{(0)}$ at $x_2^{(0)} = x_3^{(0)} = 0$. For $t < 0$, the roles of $(x_1^{(0)}, x_4^{(0)})$ and $(x_2^{(0)}, x_3^{(0)})$ are reversed.

To describe $\mathcal{G}_I$ and $\mathcal{G}_{III}$, we have to extend the $\mathbb{Z}_N$ action (4.6) on ϕ as well. Let us consider a general anzatz

$$\mathbb{Z}_N:\ (x_1, x_2, x_3, x_4; \phi) \equiv \Big(x_1, e^{2\pi i/N}x_2, e^{-2\pi i/N}x_3, x_4; \phi + \kappa\Big)\,, \tag{4.11}$$

for some constant κ that should be a multiple of $2\pi/N$. On our representatives of the U(1) equivalence classes, this action amounts to the identifications,

$$\Big[x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, x_4^{(0)}; 0\Big] \equiv \Big[e^{-i\kappa}x_1^{(0)}, e^{i\kappa + 2\pi i/N}x_2^{(0)}, e^{i\kappa - 2\pi i/N}x_3^{(0)}, e^{-i\kappa}x_4^{(0)}; 0\Big]\,. \tag{4.12}$$

Now, for $t > 0$ associated to the lift to 7 dimensions of $\mathcal{C}_I$, we know from $\mathcal{G}_I$ that the orbifold action is trivial on the S^3 base. Thus, $\kappa = 0$,

$$\mathbb{Z}_N:\ \Big[x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, x_4^{(0)}; 0\Big] \equiv \Big[x_1^{(0)}, e^{2\pi i/N}x_2^{(0)}, e^{-2\pi i/N}x_3^{(0)}, x_4^{(0)}; 0\Big]\,, \tag{4.13}$$

and we recognize the $\mathbb{R}^4/\mathbb{Z}_N$ fibration over S^3 of $\mathcal{G}_I$. Actually, to make contact with the definition (2.6), one should choose to parametrize $\mathbb{R}^4/\mathbb{Z}_N$ with $x_2^{(0)}$ and the complex conjugate $\bar{x}_3^{(0)}$ so that the $\mathbb{Z}_N$ is acting from the left. For $t < 0$, we have instead a lens space $S^3/\mathbb{Z}_N$ as expected for $\mathcal{G}_{III}$. Here also, to reproduce the $\mathbb{Z}_N$ left action convention in definition (2.8), the lens space should be parametrized by $x_2^{(0)}$ and $\bar{x}_3^{(0)}$.

What we have seen is that the 2-cycle flop $t \to -t$ in the σ-models of $\mathcal{C}_I$ and $\mathcal{C}_{III}$ can be lifted to a 3-cycle flop in the 7-dimensional σ-models of $\mathcal{G}_I$ and $\mathcal{G}_{III}$. This point of view involves the $S^1 \simeq \mathrm{U}(1)_{2,L}$ fibration introduced in section 2. As we reviewed there, $\mathcal{G}_I$ can also be obtained geometrically by lifting to M-theory the deformed conifold plus N D6-branes wrapped on S^3, by introducing $S^1 \simeq \mathrm{U}(1)_{1,L}$. However, since there is no linear σ-model description of the deformed conifold, this point of view cannot help to derive a linear σ-model description of $\mathcal{G}_I$.

On the other hand, to see explicitly in a σ-model language the 3-cycle flop between $\mathcal{G}_{II}$ and $\mathcal{G}_{III}$, it is instead the $S^1 \simeq \mathrm{U}(1)_{1,L}$ fibration over the resolved conifold with N units of RR flux through the S^2 that can be used [30]. One starts from the linear σ-model of the resolved conifold, eqs. (4.1) and (4.2), where the N units of flux are on the S^2 parametrized by x_1 and x_4 for $t > 0$, while for $t < 0$, the flux is through the flopped S^2 parametrized by x_2 and x_3. The lift to the G_2 holonomy manifolds $\mathcal{G}_{II}$ and $\mathcal{G}_{III}$ is done by adding to the D-term condition (4.2) the 2π-periodic variable ϕ' and replacing the modding action (4.1) by the identification

$$(x_1, x_2, x_3, x_4; \phi') \equiv \left(e^{i\lambda} x_1, e^{-i\lambda} x_2, e^{-i\lambda} x_3, e^{i\lambda} x_4; \phi' + N\lambda\right). \tag{4.14}$$

Gauging away ϕ' amounts to choosing $\lambda = -\phi'/N + 2k\pi/N$ with any integer k in the range $k \in \{0, \dots, N-1\}$. Representatives of these equivalence classes can thus be chosen to be of the form $[x_1'^{(0)}, x_2'^{(0)}, x_3'^{(0)}, x_4'^{(0)}; 0]$, subject to the remaining discrete $\mathbb{Z}_N$ identification

$$\mathbb{Z}_N: \left[x_1'^{(0)}, x_2'^{(0)}, x_3'^{(0)}, x_4'^{(0)}; 0\right] \equiv \left[e^{2\pi i/N} x_1'^{(0)}, e^{-2\pi i/N} x_2'^{(0)}, e^{-2\pi i/N} x_3'^{(0)}, e^{2\pi i/N} x_4'^{(0)}; 0\right], \tag{4.15}$$

and the D-term condition

$$|x_1'^{(0)}|^2 + |x_4'^{(0)}|^2 - |x_2'^{(0)}|^2 - |x_3'^{(0)}|^2 = t\,. \tag{4.16}$$

We choose to associate the phases $t > 0$ to $\mathcal{G}_{II}$ and $t < 0$ to $\mathcal{G}_{III}$. We can make contact with the definition (2.9) of $\mathcal{G}_{III}$ by noting that the $\mathbb{Z}_N$ acts on the right both on the S^3 spanned by $x_2'^{(0)}$ and $\bar{x}_3'^{(0)}$ and on $\mathbb{R}^4$ parametrized by $\bar{x}_1'^{(0)}$ and $x_4'^{(0)}$.

We now have two different descriptions of $\mathcal{G}_{III}$, the first one in eqs. (4.10) and (4.13) for $t < 0$, and the second in eqs. (4.16) and (4.15) for $t < 0$. To see explicitly that these descriptions are equivalent, we want to give the coordinate transformations that map them into each other. In the first description, $\mathcal{G}_{III}$ is covered by two charts ${x_2}^{(0)} \neq 0$ and ${x_3}^{(0)} \neq 0$, due to eq. (4.10). When ${x_2}^{(0)} \neq 0$, the phase $\arg({x_2}^{(0)})$ is well defined, so that we can introduce new coordinates $x_j'^{(0)} = x_j^{(0)} e^{in_j \arg({x_2}^{(0)})}$ with $(n_1, \dots, n_4) = (1, -2, 0, 1)$. The $\mathbb{Z}_N$ action (4.13) on the old coordinates then implies indeed the action (4.15) on the new ones. The inverse coordinate change is $x_j^{(0)} = x_j'^{(0)} e^{in_j \arg(x_2'^{(0)})}$. In the chart ${x_3}^{(0)} \neq 0$, the phase $\arg({x_3}^{(0)})$ is well defined and the analogous coordinate transformations are $x_j'^{(0)} = x_j^{(0)} e^{-in_j \arg({x_3}^{(0)})}$, whose inverse is $x_j^{(0)} = x_j'^{(0)} e^{in_j \arg(x_3'^{(0)})}$.

Similar invertible coordinate transformations that map the $\mathbb{Z}_N$ actions into each other don't exist for $t > 0$, as it should, since in this case the two models are associated to the different spaces $\mathcal{G}_I$ and $\mathcal{G}_{II}$.

JHEP11(2004)033

Acknowledgments

We are grateful to I. Antoniadis, I. Bakas, I. Brunner, P. Mayr and S. Theisen for useful discussions. We acknowledge the CERN theory division, where part of this work has been done. While at Ecole Polytechnique, P.K. was supported by an European Commission Marie Curie postdoctoral fellowship under the contract number HPMF-CT-2000-00919. The work of H.P. is supported in part by the European networks HPRN-CT-2000-00148, MRTN-CT-2004-503369, MRTN-CT-2004-005104 and the E.C. Excellence Grant MEXT-CT-2003-509661.

A. The G_2 holonomy metrics

In this appendix we review the construction of G_2 metrics on $\mathrm{Spin}(S^3)$, the idea of which we have already recalled around eq. (2.1). We discuss their isometry groups and in particular the action of their subgroups (2.5) along which we want to perform KK reductions.

We represent $g \in \mathrm{SU}(2) \simeq S^3$ either as

$$g = \begin{pmatrix} Z & iZ' \\ i\bar{Z}' & \bar{Z} \end{pmatrix}, \qquad \text{with } Z, Z' \in \mathbb{C} \qquad \text{and} \qquad \det(g) = |Z|^2 + |Z'|^2 = 1\,, \tag{A.1}$$

or introduce the Euler angles $\theta \in (0, \pi)$, $\varphi \in [0, 2\pi)$ and $\psi \in [0, 4\pi)$ as coordinates on the open complement of the set $\{Z = 0\} \cup \{Z' = 0\}$,

$$g = \exp\left(\frac{i}{2}\psi\sigma_3\right) \exp\left(\frac{i}{2}\theta\sigma_1\right) \exp\left(\frac{i}{2}\varphi\sigma_3\right) = \begin{pmatrix} \cos(\theta/2)e^{i(\psi+\varphi)/2} & i\sin(\theta/2)e^{i(\psi-\varphi)/2} \\ i\sin(\theta/2)e^{-i(\psi-\varphi)/2} & \cos(\theta/2)e^{-i(\psi+\varphi)/2} \end{pmatrix}, \tag{A.2}$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

denote the Pauli matrices. Define

$$\Theta_j := g_j^{-1} dg_j = \frac{i}{2}\omega^a_{j,L}\sigma_a\,, \qquad (j = 1, 2, 3) \tag{A.3}$$

where $\omega^a_{j,L}$ are the three left-invariant one-forms on the j-th copy of SU(2) in (2.1),

$$\begin{aligned} w^1_{j,L} &= \cos(\varphi_j)d\theta_j + \sin(\varphi_j)\sin(\theta_j)d\psi_j\,, \\ w^2_{j,L} &= \sin(\varphi_j)d\theta_j - \cos(\varphi_j)\sin(\theta_j)d\psi_j\,, \\ w^1_{j,L} &= d\varphi_j + \cos(\theta_j)d\psi_j\,. \end{aligned} \tag{A.4}$$

In [16] the G_2 metric is given in variables

$$\begin{aligned} da^2 &:= -\operatorname{Tr}\left(\Theta_2 \otimes \Theta_2 + \Theta_3 \otimes \Theta_3 - \Theta_2 \otimes \Theta_3 - \Theta_3 \otimes \Theta_2\right), \\ db^2 &:= -\operatorname{Tr}\left(\Theta_1 \otimes \Theta_1 + \Theta_3 \otimes \Theta_3 - \Theta_1 \otimes \Theta_3 - \Theta_3 \otimes \Theta_1\right), \\ dc^2 &:= -\operatorname{Tr}\left(\Theta_1 \otimes \Theta_1 + \Theta_2 \otimes \Theta_2 - \Theta_1 \otimes \Theta_2 - \Theta_2 \otimes \Theta_1\right) \end{aligned} \tag{A.5}$$

JHEP11(2004)033

as

$$ds^2 = \frac{dr \otimes dr}{1-(r_0/r)^3} + \frac{r^2\left(1-(r_0/r)^3\right)}{72}\left(2da^2 - db^2 + 2dc^2\right) + \frac{r^2}{24}db^2\,, \qquad \text{(A.6)}$$

where $r \in [r_0, \infty)$, with r_0 being a modulus of the solution describing the radius of the base S^3 of $\mathrm{Spin}(S^3)$. The three group elements g_j are subject to the identification of eq. (2.1), which we can use e.g. to fix $g_3 \equiv 1$, bringing the above metric to the form,

$$ds^2 = \frac{dr \otimes dr}{1-(r_0/r)^3} - \frac{r^2\left(1-(r_0/r)^3\right)}{72}\,\mathrm{Tr}\left[(2\Theta_2 - \Theta_1)\otimes(2\Theta_2-\Theta_1)\right] - \frac{r^2}{24}\,\mathrm{Tr}\left(\Theta_1 \otimes \Theta_1\right). \qquad \text{(A.7)}$$

Since

$$-\,\mathrm{Tr}\left(\Theta_j \otimes \Theta_k\right) = \frac{1}{2}\sum_{a=1}^{3}\left(\omega^a_{j,L}\otimes\omega^a_{k,L}\right),$$

where in particular

$$\begin{aligned} -\frac{1}{2}\,\mathrm{Tr}(\Theta_1\otimes\Theta_1) &= \frac{1}{4}\sum_{a=1}^{3}(\omega^a_{1,L}\otimes\omega^a_{1,L}) \\ &= \frac{1}{4}\Big(d\theta_1\otimes d\theta_1 + d\varphi_1\otimes d\varphi_1 + d\psi_1\otimes d\psi_1 + \\ &\qquad + \cos(\theta_1)[d\varphi_1\otimes d\psi_1 + d\psi_1 \otimes d\varphi_1]\Big) \qquad \text{(A.8)} \\ &= \frac{1}{2}\left(dZ_1\otimes d\bar{Z}_1 + d\bar{Z}_1\otimes dZ_1 + dZ_1'\otimes d\bar{Z}_1' + d\bar{Z}_1'\otimes dZ_1'\right)\Big|_{\{|Z_1|^2+|Z_1'|^2=1\}} \end{aligned}$$

is the standard metric on the three-sphere inherited from its embedding into euclidean $\mathbb{R}^4$, the metric (A.7) indeed agrees with the complete G_2 holonomy metric originally given in [25]. In this example, the sphere parametrized by g_1 becomes the base of the fibration, whereas the one associated to g_2 is "filled" and turned into $\mathbb{R}^4$. Thus, the metric (A.7) corresponds to the manifold we call $\mathcal{G}_{II}$ in the text, once we take into account the relevant $\mathbb{Z}_N$ isometry orbifold action.

Note that the $\mathrm{SU}(2)_{1,L}\times\mathrm{SU}(2)_{2,L}\times\mathrm{SU}(2)_{3,L}$ isometry group (2.3) of (A.6), after gauge fixing $g_3\equiv 1$, is identified with an $\mathrm{SU}(2)_{1,L}\times\mathrm{SU}(2)_{2,L}\times[\mathrm{SU}(2)_{1,R}\times\mathrm{SU}(2)_{2,R}]_D$ isometry group of (A.7), where the subindex D denotes the diagonal subgroup,

$$(g_1, g_2, 1) \to (h_1 g_1 h_3^{-1}, h_2 g_2 h_3^{-1}, 1)\,,$$

where

$$(h_1, h_2, h_3) \in \mathrm{SU}(2)_{1,L}\times\mathrm{SU}(2)_{2,L}\times\mathrm{SU}(2)_{3,L}\,. \qquad \text{(A.9)}$$

We are particularly interested in the action of U(1) isometry subgroups of some $\mathrm{SU}(2)_{j,L}$ represented by matrix multiplication of

$$\xi(\omega) = \begin{pmatrix} e^{i\omega} & 0 \\ 0 & e^{-i\omega}\end{pmatrix} \in \mathrm{SU}(2)_{j,L}\,. \qquad \text{(A.10)}$$

JHEP11(2004)033

On the one hand, the left-action $m_L(\xi(\omega), g) = \xi(\omega) g$ acts as

$$m_L \; : \begin{cases} (Z, Z') \mapsto (e^{i\omega} Z, e^{i\omega} Z') \\ \qquad\qquad \text{or} \\ (\theta, \varphi, \psi) \mapsto (\theta, \varphi, \psi + 2\omega) \,, \end{cases} \tag{A.11}$$

where we can periodically continue the range of the Euler angle ψ, i.e. define it modulo 4π. On the other hand, the right-action $m_R(\xi(\omega), g) = g\xi(\omega)$ acts as

$$m_R : \begin{cases} (Z, Z') \mapsto (e^{i\omega} Z, e^{-i\omega} Z') \\ \qquad\qquad \text{or} \\ (\theta, \varphi, \psi) \mapsto \begin{cases} (\theta\,, \varphi + 2\omega \quad\;\;\, , \psi \qquad\;\;) \,, & \text{for } \quad 0 \qquad\; \le \omega < \; \pi - \varphi/2 \,, \\ (\theta\,, \varphi + 2\omega - 2\pi\,, \psi + 2\pi) \,, & \text{for } \; \pi - \varphi/2 \; \le \omega < 2\pi - \varphi/2 \,, \\ (\theta\,, \varphi + 2\omega - 4\pi\,, \psi \qquad\;\;) \,, & \text{for } 2\pi - \varphi/2 \le \omega < \quad\;\; 2\pi \,. \end{cases} \end{cases} \tag{A.12}$$

To obtain $\mathcal{G}_{II}$, one considers the modding action of the $\mathbb{Z}_N$ subgroup of $\mathrm{SU}(2)_{1,L}$, which thus identifies $\psi_1 \equiv \psi_1 + 4\pi/N$. The base in $\mathcal{G}_{II}$ is then the lens space we shall denote $L_N := S^3/m_L(\mathbb{Z}_N)$. The metrics on $\mathcal{G}_I$ and $\mathcal{G}_{III}$ are constructed analogously.

B. Kaluza-Klein reduction of the lens space $L_N := S^3/m_L(\mathbb{Z}_N)$

We want to recall in this appendix how the Kaluza-Klein reduction of $\mathcal{G}_{II}$ and $\mathcal{G}_{III}$ along the orbit of the isometry $\mathrm{U}(1)_{1,L}$ results in a geometry that is a fibration over S^2, with N units of RR two-form flux through S^2. This amounts to concentrate in particular on the Hopf reduction of the lens space $L_N := S^3/m_L(\mathbb{Z}_N)$.

In general, when performing the Kaluza-Klein reduction of an eleven-dimensional metric in M-theory (in the Einstein frame)

$$d\hat{s}^2 = \sum_{\hat{\mu},\hat{\nu}=0}^{10} \hat{g}_{\hat{\mu}\hat{\nu}}\, dx^{\hat{\mu}} \otimes dx^{\hat{\nu}} \,,$$

where $x^{10} \in [0, 2\pi)$ parametrizes the orbit of a U(1) isometry, one identifies the metric $ds^2 = \sum_{\mu,\nu=0}^{9} g_{\mu\nu}\, dx^\mu \otimes dx^\nu$, RR one-form gauge potential $A = A_\mu dx^\mu$ and dilaton Φ in type IIA via

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} e^{-2\Phi/3} g_{\mu\nu} + e^{4\Phi/3} A_\mu A_\nu & e^{4\Phi/3} A_\mu \\ e^{4\Phi/3} A_\nu & e^{4\Phi/3} \end{pmatrix} . \tag{B.1}$$

Assuming that $g_{\mu\nu}$, A_μ and Φ do not depend on x^{10}, the 11-dimensional Einstein-Hilbert action reduces to the 10-dimensional Einstein-Hilbert-Maxwell action (in the string frame). The gauge potential A obtained this way may be well-defined only upon performing a gauge transformation, but its curvature $F = dA$ will in general be well-defined.

$\mathcal{G}_{II}$ and $\mathcal{G}_{III}$ are $\mathbb{R}^4$-fibrations over $L_N := S^3/m_L(\mathbb{Z}_N)$, where the base can be identified with the SU(2) parametrized by g_1 in eqs. (2.7) and (2.8) and modded by the left-action of its $\mathbb{Z}_N$ subgroup. When seeing this $\mathrm{SU}(2) \simeq S^3$ itself as a Hopf fibration, $H \,:\, S^3 \to S^2$, over a two-sphere spanned by (θ_1, φ_1) via

$$X_1 = 2\,\mathrm{Re}(Z_1 \bar{Z}_1') = Z_1 \bar{Z}_1' + \bar{Z}_1 Z_1' = \sin(\theta_1)\cos(\varphi_1) \,,$$

$$
\begin{aligned}
X_1' &= 2\,\mathrm{Im}(Z_1\bar{Z}_1') = -i(Z_1\bar{Z}_1' - \bar{Z}_1 Z_1') = \sin(\theta_1)\sin(\varphi_1)\,, \\
X_1'' &= |Z_1|^2 - |Z_1'|^2 = Z_1\bar{Z}_1 - Z_1'\bar{Z}_1' = \cos(\theta_1)\,,
\end{aligned}
\tag{B.2}
$$

the fiber parametrized by ψ_1 is generated by the orbit of the left-action (A.11) of $\mathrm{U}(1)_{1,L}$ on $S^3 \simeq \mathrm{SU}(2)$, the base being invariant under this action. Hence L_N and thus $\mathcal{G}_{II}$ and $\mathcal{G}_{III}$ are fibrations over S^2.

In order to determine the RR two-form flux through the S^2, we can restrict to the base part

$$
\begin{aligned}
d\hat{s}^2 &= -\frac{R^2}{24}\,\mathrm{Tr}\,(\Theta_1 \otimes \Theta_1) \\
&= \frac{R^2}{48}\left(d\theta_1 \otimes d\theta_1 + d\varphi_1 \otimes d\varphi_1 + \frac{4}{N^2} d\tilde{\psi}_1 \otimes d\tilde{\psi}_1 + \frac{2}{N}\cos(\theta_1)[d\varphi_1 \otimes d\tilde{\psi}_1 + d\tilde{\psi}_1 \otimes d\varphi_1]\right)
\end{aligned}
$$

of (A.7), where $\tilde{\psi}_1 = N\psi_1/2 \in [0, 2\pi)$ and $R^2 = r^2\left(1 + \frac{1-(r_0/r)^3}{3}\right)$. With (B.1) we read off,

$$
e^{4\Phi/3} = \frac{R^2}{12N^2}\,, \quad ds^2 = \frac{R^3}{96\sqrt{3}N}(d\theta_1 \otimes d\theta_1 + \sin^2(\theta_1) d\varphi_1 \otimes d\varphi_1)\,, \quad A = \frac{N}{2}\cos(\theta_1) d\varphi_1\,.
\tag{B.3}
$$

From the curvature

$$
F = dA = -\frac{N}{2}\sin(\theta_1)\,d\theta_1 \wedge d\varphi_1\,,
$$

one computes as first Chern class of the bundle $H' :\ L_N \to S^2$ defined as in eq. (B.2),

$$
c_1(H') = \frac{1}{2\pi}\int_{S^2} F = -\frac{N}{4\pi}\int_{\varphi_1=0}^{2\pi}\int_{\theta_1=0}^{\pi}\sin(\theta_1)\,d\theta_1 \wedge d\varphi_1 = -N\,,
\tag{B.4}
$$

which also gives the units of RR two-form flux through the S^2.

C. Kaluza-Klein reduction of $(S^3 \times \mathbb{R}^4)/m_R(\mathbb{Z}_N)$

In this appendix, we want to perform the Kaluza-Klein reduction of $\mathcal{G}_{III}$ along the orbits of $\mathrm{U}(1)_{2,L}$. In the case where we have used the identification (2.1) to fix the gauge $g_1 \equiv 1$ in (2.9), the orbifold action turns into a simultaneous *right*-action of $\mathbb{Z}_N$ on g_2 and g_3, as explained in appendix A. This action is free due to the noncontractibility of the S^3 parametrized by g_2. The metric on $\mathcal{G}_{III}$ looks like (A.7) with $\Theta_1 \mapsto \Theta_2$ and $\Theta_2 \mapsto \Theta_3$. In addition, we identify points that are mapped into each other by the simultaneous right-action action (A.12) on g_2 and g_3 for $\omega = 2\pi k/N$, with $k = 0, \ldots, N-1$. In order to analyze the consequences for the Kaluza-Klein reduction, we distinguish the cases of odd respectively even N.

N odd: consider first generic points such that $Z_2, Z_2' \neq 0$, so that we can use Euler angles $\theta_2, \varphi_2, \psi_2$ on the noncontractible three-sphere, whereas we use complex coordinates Z_3, Z_3' on the $\mathbb{R}^4$-fiber. Let $\varphi_2 \in [0, 2\pi/N)$. Then the $\mathbb{Z}_N$ right action (A.12) identifies the points

$$
(\theta_2, \varphi_2, \psi_2, Z_3, Z_3') \equiv \left(\theta_2, \varphi_2 + \frac{2\pi k}{N}, \psi_2 + 2\pi k \bmod 4\pi, (-1)^k e^{i\pi k/N} Z_3, (-1)^k e^{-i\pi k/N} Z_3'\right),
\tag{C.1}
$$

for $k = 0, \ldots, N-1$. This action is always free on the noncontractible S^3 parametrized by $(\theta_2, \varphi_2, \psi_2)$.

Performing the KK reduction (B.1) [with $\tilde{\psi}_2 = \psi_2/2 \in [0, 2\pi)$] means integrating over the orbits of the isometry $\psi_2 \to \psi_2 + 2\omega$ with ψ_2 defined modulo 4π. On the resulting six-dimensional space, we have to identify the points

$$(\theta_2, \varphi_2, Z_3, Z_3') \equiv \left(\theta_2, \varphi_2 + \frac{2\pi k}{N}, (-1)^k e^{i\pi k/N} Z_3, (-1)^k e^{-i\pi k/N} Z_3'\right), \tag{C.2}$$

where $k = 0, \ldots, N-1$ and again φ_2 is taken to lie in the range $[0, 2\pi/N)$. This results in a space of topology

$$(S^2 \times \mathbb{R}^4)/\mathbb{Z}_N, \tag{C.3}$$

where the $\mathbb{Z}_N$ acts on the two-sphere by rotations around the axis through the north and south poles by angles $2\pi k/N$, combined with an action

$$(Z_3, Z_3') \equiv \left((-1)^k e^{i\pi k/N} Z_3, (-1)^k e^{-i\pi k/N} Z_3'\right)$$

on $\mathbb{R}^4$, for $k = 0, \ldots, N-1$, which generates the familiar action (A.12) for $\mathbb{Z}_N$ on $\mathbb{R}^4$ when running through all values of k. Note that on the six-dimensional space, the orbifold action now has two fixed points, namely the poles of the base S^2, where either $Z_2 = Z_3 = Z_3' = 0$ or $Z_2' = Z_3 = Z_3' = 0$. Since we had previously excluded these loci, let's discuss them now.

Consider the five-dimensional sublocus $\{Z_2' = 0\} \simeq S^1 \times \mathbb{R}^4$ inside $\mathcal{G}_{III}$ and let $Z_2 = e^{i\tilde{\varphi}_2}$. The $\mathbb{Z}_N$ orbifold acts on it as

$$(\tilde{\varphi}_2, Z_3, Z_3') \equiv \left(\tilde{\varphi}_2 + \frac{2\pi k}{N}, e^{2\pi i k/N} Z_3, e^{-2\pi i k/N} Z_3'\right), \tag{C.4}$$

where $k = 0, \ldots, N-1$. The KK reduction is along the S^1 parametrized by $\tilde{\varphi}_2$ resulting in the sublocus $\{\text{south pole}\} \times \mathbb{R}^4$ with the familiar $\mathbb{Z}_N$ action on $\mathbb{R}^4$. The discussion of the sublocus $\{Z_2 = 0\}$ is completely analogous upon replacing $\tilde{\varphi}_2$ by $-\tilde{\varphi}_2'$ and the south pole by the north pole.

Over each point on the six-dimensional space, there was a U(1)-orbit that gives rise to a U(1) gauge connection A. Over a generic point, the orbifold group doesn't act within the gauge fiber over this point — it merely gives a relative half-twist (depending on k mod 2) between the disjoint orbits over its N pre-image points. Over the fixed points, i.e. the north and south poles of the S^2 (times the origin in $\mathbb{R}^4$), on the contrary, the orbifold group $\mathbb{Z}_N$ acts within the gauge fibers over these two points. We interpret the behavior at generic points as indicating one unit of RR two-form flux through the S^2, since performing the reduction (B.1) for the G_2 metric on $\mathcal{G}_{III}$ [with $\tilde{\psi}_2 = \psi_2/2 \in [0, 2\pi)$] leads to

$$\begin{aligned}
e^{4\Phi/3} &= \frac{1}{36r}(4r^3 - r_0^3), \\
A &= \frac{1}{2}\cos(\theta_2) d\varphi_2 - \frac{r^3 - r_0^3}{(4r^3 - r_0^3)}\Big(\cos(\theta_2) d\varphi_3 + \sin(\theta_2)\sin(\varphi_2 - \varphi_3) d\theta_3 + \\
&\qquad + [\cos(\theta_2)\cos(\theta_3) + \cos(\varphi_2 - \varphi_3)\sin(\theta_2)\sin(\theta_3)] d\psi_3\Big)
\end{aligned}$$

and gives one unit of RR two-form flux through the covering S^2 parametrized by $\varphi_2 \in [0, 2\pi)$ and $\theta_2 \in (0, \pi)$,

$$\frac{1}{2\pi} \int_{S^2} dA = -1 .$$

The additional $\mathbb{Z}_N$ action within the U(1) gauge fiber over the poles is interpreted as shifting the masses of string states that arise in the twisted sectors associated to these fixed points. This mechanism was introduced in [20] in order to describe the type-IIA dual of the six-dimensional CHL compactification. It may also be thought of as additional flux localized at the fixed points.

N even: again we consider first generic points such that $Z_2, Z_2' \neq 0$, and let $k = l + \frac{N}{2}\alpha$, with $l = 0, \ldots, N/2 - 1$ and $\alpha = 0, 1$. For $\varphi_2 \in [0, 4\pi/N)$, the $\mathbb{Z}_N$ right action (A.12) identifies all the points with coordinates

$$(\theta_2, \varphi_2, \psi_2, Z_3, Z_3') \equiv \left(\theta_2, \varphi_2 + \frac{2\pi l}{N/2}, \psi_2 + 2\pi\alpha \bmod 4\pi, (-1)^\alpha e^{2\pi i l/N} Z_3, (-1)^\alpha e^{-2\pi i l/N} Z_3'\right), \tag{C.5}$$

for $l = 0, \ldots, N/2 - 1$ and $\alpha = 0, 1$. Again this action is always free on the noncontractible S^3 parametrized by $(\theta_2, \varphi_2, \psi_2)$.

Performing the KK reduction (B.1) [with $\tilde{\psi}_2 = \psi_2/2 \in [0, 2\pi)$] means integrating over the orbits of the isometry $\psi_2 \to \psi_2 + 2\omega$, with ψ_2 defined modulo 4π. On the resulting six-dimensional space, we have to identify the points

$$(\theta_2, \varphi_2, Z_3, Z_3') \equiv \left(\theta_2, \varphi_2 + \frac{2\pi l}{N/2}, (-1)^\alpha e^{2\pi i l/N} Z_3, (-1)^\alpha e^{-2\pi i l/N} Z_3'\right), \tag{C.6}$$

where $l = 0, \ldots, N/2 - 1$ and $\alpha = 0, 1$ and again φ_2 is taken to lie in the range $[0, 4\pi/N)$. This results in a space of topology

$$\left(S^2 \times \mathbb{R}^4\right) / (\mathbb{Z}_2 \times \mathbb{Z}_{N/2}) , \tag{C.7}$$

where the $\mathbb{Z}_2$ acts trivially on the two-sphere and the $\mathbb{Z}_{N/2}$ by rotations around the axis through the north and south poles by angles $4\pi l/N$, whereas the $\mathbb{Z}_2 \times \mathbb{Z}_{N/2}$ acts on the $\mathbb{R}^4$ as

$$(Z_3, Z_3') \equiv \left(e^{2i\pi(l+\frac{N}{2}\alpha)/N} Z_3, e^{-2i\pi(l+\frac{N}{2}\alpha)/N} Z_3'\right) = \left(e^{2i\pi k/N} Z_3, e^{-2i\pi k/N} Z_3'\right),$$

with $k = l + \frac{N}{2}\alpha$, where $l = 0, \ldots, \frac{N}{2} - 1$ and $\alpha = 0, 1$. Note that the generator of $\mathbb{Z}_2$ always leads to an $S^2/\mathbb{Z}_{N/2}$ of A_1 singularities given by $Z_3 = Z_3' = 0$. The poles of the base, where in addition either $Z_2 = 0$ or $Z_2' = 0$, are fixed by the whole $\mathbb{Z}_N$. The discussion of these two fixed points and their $\mathbb{R}^4$ fibers is word for word the same as the one around (C.4) for odd N.

Over a generic point of the six-dimensional space, the orbifold group doesn't act within the U(1) gauge fiber over this point — it merely gives a relative half-twist (depending on α) between the disjoint orbits over its N pre-image points. Over the $\mathbb{Z}_2$-fixed base $S^2/\mathbb{Z}_{N/2}$, however, these half-twists generate a $\mathbb{Z}_2$ action within the gauge fiber over it. At the $\mathbb{Z}_N$-fixed points (the poles of the S^2 times the origin in $\mathbb{R}^4$) there is a localized embedding of the $\mathbb{Z}_N$-action within the gauge fiber over them. By the same calculation as for odd N, we interpret the behavior at generic points as indicating one unit of RR two-form flux through the S^2 parametrized by (θ_2, φ_2). The additional $\mathbb{Z}_2$ (respectively $\mathbb{Z}_N$) action localized within the U(1) gauge fiber over the $\mathbb{Z}_2$ (respectively $\mathbb{Z}_N$)-fixed locus implies that these fixed sets don't lead to new massless states through their associated twisted sectors. In particular there are no massless vector bosons associated to the vanishing cycle of the A_1 singularity at the $\mathbb{Z}_2$-fixed sphere.

This mechanism of removing massless states in type-IIA string theory from the twisted sectors associated to the fixed point set of an orbifold action by a localized embedding of this action also in the U(1) gauge bundle of the RR one-form potential is exactly what was also used in [20] to construct a type-IIA dual of the six-dimensional CHL compactification. It would be interesting to better understand this mechanism in the type-IIA setting but unfortunately we don't know the CFT description of this nontrivial RR-background. It is, however, very reminiscent of a familiar shift orbifold on an additional circle describing the gauge bundle.

References

[1] C.M. Hull and P.K. Townsend, *Unity of superstring dualities*, *Nucl. Phys.* **B 438** (1995) 109 [hep-th/9410167].

[2] E. Witten, *String theory dynamics in various dimensions*, *Nucl. Phys.* **B 443** (1995) 85 [hep-th/9503124].

[3] C.M. Hull and P.K. Townsend, *Enhanced gauge symmetries in superstring theories*, *Nucl. Phys.* **B 451** (1995) 525 [hep-th/9505073].

[4] P.S. Aspinwall, *Enhanced gauge symmetries and K3 surfaces*, *Phys. Lett.* **B 357** (1995) 329 [hep-th/9507012].

[5] S. Kachru and C. Vafa, *Exact results for $N = 2$ compactifications of heterotic strings*, *Nucl. Phys.* **B 450** (1995) 69 [hep-th/9505105].

[6] S. Ferrara, J.A. Harvey, A. Strominger and C. Vafa, *Second quantized mirror symmetry*, *Phys. Lett.* **B 361** (1995) 59 [hep-th/9505162].

[7] S. Kachru, A. Klemm, W. Lerche, P. Mayr and C. Vafa, *Nonperturbative results on the point particle limit of $N = 2$ heterotic string compactifications*, *Nucl. Phys.* **B 459** (1996) 537 [hep-th/9508155].

[8] I. Antoniadis and H. Partouche, *Exact monodromy group of $N = 2$ heterotic superstring*, *Nucl. Phys.* **B 460** (1996) 470 [hep-th/9509009].

[9] A. Klemm and P. Mayr, *Strong coupling singularities and non-abelian gauge symmetries in $N = 2$ string theory*, *Nucl. Phys.* **B 469** (1996) 37 [hep-th/9601014].

[10] T.R. Taylor and C. Vafa, *RR flux on Calabi-Yau and partial supersymmetry breaking*, *Phys. Lett.* **B 474** (2000) 130 [hep-th/9912152].

[11] P. Mayr, *On supersymmetry breaking in string theory and its realization in brane worlds*, *Nucl. Phys.* **B 593** (2001) 99 [hep-th/0003198].

JHEP11(2004)033

[12] I. Antoniadis, H. Partouche and T.R. Taylor, *Spontaneous breaking of $N = 2$ global supersymmetry*, *Phys. Lett.* **B 372** (1996) 83 [hep-th/9512006].

[13] H. Partouche and B. Pioline, *Partial spontaneous breaking of global supersymmetry*, *Nucl. Phys.* **56B** *(Proc. Suppl.)* (1997) 322 [hep-th/9702115].

[14] C. Vafa, *Superstrings and topological strings at large N*, *J. Math. Phys.* **42** (2001) 2798 [hep-th/0008142].

[15] M. Atiyah, J.M. Maldacena and C. Vafa, *An M-theory flop as a large-N duality*, *J. Math. Phys.* **42** (2001) 3209 [hep-th/0011256].

[16] M. Atiyah and E. Witten, *M-theory dynamics on a manifold of G_2 holonomy*, *Adv. Theor. Math. Phys.* **6** (2003) 1 [hep-th/0107177].

[17] M. Aganagic and C. Vafa, *Mirror symmetry and a G_2 flop*, *J. High Energy Phys.* **05** (2003) 061 [hep-th/0105225].

[18] A. Giveon, A. Kehagias and H. Partouche, *Geometric transitions, brane dynamics and gauge theories*, *J. High Energy Phys.* **12** (2001) 021 [hep-th/0110115].

[19] K. Fujiwara, H. Itoyama and M. Sakaguchi, *Supersymmetric U(N) gauge model and partial breaking of $N = 2$ supersymmetry*, hep-th/0409060.

[20] J.H. Schwarz and A. Sen, *Type-IIA dual of the six-dimensional CHL compactification*, *Phys. Lett.* **B 357** (1995) 323 [hep-th/9507027].

[21] B.S. Acharya, *On realising $N = 1$ super Yang-Mills in M-theory*, hep-th/0011089.

[22] P. Kaste, A. Kehagias and H. Partouche, *Phases of supersymmetric gauge theories from M-theory on G_2 manifolds*, *J. High Energy Phys.* **05** (2001) 058 [hep-th/0104124].

[23] H. Partouche and B. Pioline, *Rolling among G_2 vacua*, *J. High Energy Phys.* **03** (2001) 005 [hep-th/0011130].

[24] R. Bryant and S. Salomon, *On the construction of some complete metrics with exceptional holonomy*, *Duke Math. J.* **58** (1989) 829.

[25] G.W. Gibbons, D.N. Page and C.N. Pope, *Einstein metrics on S^3, $\mathbb{R}^3$ and $\mathbb{R}^4$ bundles*, *Commun. Math. Phys.* **127** (1990) 529.

[26] A. Brandhuber, J. Gomis, S.S. Gubser and S. Gukov, *Gauge theory at large-N and new G_2 holonomy metrics*, *Nucl. Phys.* **B 611** (2001) 179 [hep-th/0106034].

[27] A. Sen, *A note on enhanced gauge symmetries in M- and string theory*, *J. High Energy Phys.* **09** (1997) 001 [hep-th/9707123].

[28] P. Candelas, P.S. Green and T. Hubsch, *Rolling among Calabi-Yau vacua*, *Nucl. Phys.* **B 330** (1990) 49.

[29] E. Witten, *Phases of $N = 2$ theories in two dimensions*, *Nucl. Phys.* **B 403** (1993) 159 [hep-th/9301042].

[30] M. Aganagic and C. Vafa, *G_2 manifolds, mirror symmetry and geometric engineering*, hep-th/0110171.

JHEP11(2004)033